

SEMESTER 1
APPLIED MATHEMATICS SOLVED PAPER – MAY 2018

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1(a) If $\tan \frac{x}{2} = \tanh \frac{u}{2}$, show that $u = \log[(\tan(\frac{\pi}{4} + \frac{x}{2}))]$ [3]

Ans : Given that : $\tan \frac{x}{2} = \tanh \frac{u}{2}$

$$\frac{u}{2} = \tanh^{-1}[\tan \frac{x}{2}]$$

$$\therefore u = 2 \tanh^{-1}[\tan \frac{x}{2}]$$

By using Inverse hyperbolic function,

$$= \log \left[\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right]$$

$$\text{But } \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} = \frac{\frac{\pi}{4} + \tan \frac{x}{2}}{\frac{\pi}{4} - \tan \frac{x}{2}} = \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$\therefore u = \log \left[(\tan(\frac{\pi}{4} + \frac{x}{2})) \right]$$

Hence proved.

(b) Prove that the following matrix is orthogonal & hence find A^{-1} . [3]

$$A = \frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Ans : Let $A = \frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

Transpose of A is given by ,

$$A^T = \frac{1}{3} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

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$$\begin{aligned} A \cdot A^T &= \frac{1}{9} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{I}{9} \end{aligned}$$

The given matrix A is orthogonal.

The inverse of an orthogonal matrix is always equal to the Transpose of that particular matrix.

$$\therefore A^{-1} = \frac{1}{3} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

(c) State Euler's theorem on homogeneous function of two variables

& if $u = \frac{x+y}{x^2+y^2}$ then evaluate $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$ [3]

Ans : Euler's theorem : If a function 'u' is homogeneous with degree 'n' then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$$

Let $u = \frac{x+y}{x^2+y^2}$

Put $x = xt$ and $y = yt$

$$\begin{aligned} F(x,y) &= \frac{xt+yt}{(xt)^2+(yt)^2} = \frac{1}{t} \left[\frac{x+y}{x^2+y^2} \right] \\ &= t^{-1} \cdot f(u) \end{aligned}$$

Hence the given function 'u' is homogeneous with degree $n=-1$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = - \left[\frac{x+y}{x^2+y^2} \right]$$

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(d) If $u = r^2 \cos 2\theta, v = r^2 \sin 2\theta$. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$. [3]

Ans : $u = r^2 \cos 2\theta$ $v = r^2 \sin 2\theta$

Diff. u and v w.r.t r and θ partially to apply it in jacobian

$$\begin{aligned} \frac{\partial(u,v)}{\partial(r,\theta)} &= \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r \cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix} \\ &= 4r^3 \cos^2 2\theta + 4r^3 \sin^2 2\theta \\ &= 4r^3 (\cos^2 2\theta + \sin^2 2\theta) \end{aligned}$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$$

(e) Find the nth derivative of $\cos 5x \cdot \cos 3x \cdot \cos x$. [4]

Ans : let $y = \cos 5x \cdot \cos 3x \cdot \cos x$

$$= \frac{\cos(5x-3x) + \cos(5x+3x)}{2} \cos x \left\{ \cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \right\}$$

$$= \frac{1}{2} [\cos 2x \cdot \cos x + \cos 8x \cdot \cos x]$$

$$y = \frac{1}{4} [\cos 3x + \cos x + \cos 9x + \cos 7x]$$

Take n th derivative,

n th derivative of $\cos(ax + b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

$$y_n = \frac{1}{4} [3^n \cos\left(\frac{n\pi}{2} + 3x\right) + \cos\left(\frac{n\pi}{2} + x\right) + 9^n \cos\left(\frac{n\pi}{2} + 9x\right) + 7^n \cos\left(\frac{n\pi}{2} + 7x\right)]$$

(f) Evaluate : $\lim_{x \rightarrow 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$ [4]

Ans : let $L = \lim_{x \rightarrow 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$

Take log on both sides,

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$$\begin{aligned}\therefore \log L &= \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{2x+1}{x+1} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{2x+1}{x^2+x} \right)\end{aligned}$$

Apply L'Hospital rule ,

$$\therefore \log L = \lim_{x \rightarrow 0} \left(\frac{2}{2x+1} \right)$$

$$\therefore \log L = 2$$

$$\therefore L = e^2$$

Q. 2(a) Solve $x^4 - x^3 + x^2 - x + 1 = 0$.

[6]

Ans : $x^4 - x^3 + x^2 - x + 1 = 0$

Multiply the given eqn by (x+1),

$$(x+1)(x^4 - x^3 + x^2 - x + 1) = 0$$

$$x^5 = (-1)$$

But $-1 = \cos \pi + i \sin \pi$

$$\therefore x = [\cos \pi + i \sin \pi]^{1/5}$$

But By De Moivres theorem ,

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

$$\therefore x = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$$

Add period $2k\pi$,

$$\therefore x = \cos(1 + 2k) \frac{\pi}{5} + i \sin(1 + 2k) \frac{\pi}{5}$$

Where $k = 0, 1, 2, 3, 4$.

The roots of given eqn is given by ,

Put $k=0$ $x_0 = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5} = e^{\pi/5}$

$k=1$ $x_1 = \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} = e^{3\pi/5}$

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$$k=2 \quad x_2 = \cos \frac{\pi}{1} + i \sin \frac{\pi}{1} = e^{\pi/1}$$

$$k=3 \quad x_3 = \cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5} = e^{7\pi/5}$$

$$k=4 \quad x_4 = \cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5} = e^{9\pi/5}$$

The roots of eqn are : $e^{\pi/5}, e^{3\pi/5}, e^{\pi/1}, e^{7\pi/5}, e^{9\pi/5}$.

(b) If $y=e^{\tan^{-1} x}$. Prove that

$$(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0 \quad [6]$$

Ans : $y = e^{\tan^{-1} x} \quad \dots\dots(1)$

Diff. w.r.t x,

$$y_1 = e^{\tan^{-1} x} \frac{1}{x^2+1}$$

$$(x^2 + 1)y_1 = e^{\tan^{-1} x} = y \quad \text{-----(from 1)}$$

Again diff. w.r.t x,

$$(x^2 + 1)y_2 + 2xy_1 = y_1 \quad \dots\dots\dots(1)$$

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$$(uv)_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + u v_n$$

$$u = (x^2 + 1), v = y_2 \quad \dots\text{for first term in eqn (1)}$$

$$u = 2x, v = y_1 \quad \dots\text{for second term in eqn (1)}$$

$$\therefore (1 + x^2)y_{n+2} + 2(n + 1)xy_{n+1} + n(n + 1)y_n - y_{n+1} = 0$$

$$\therefore (1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$$

Hence Proved.

(c) Examine the function $f(x, y) = xy(3 - x - y)$ for extreme values & find maximum and minimum values of $f(x, y)$. [8]

Ans : $f(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2$

Diff. function w.r.t x and y partially,

$$\frac{\partial f(x,y)}{\partial x} = 3y - 2xy - y^2 \qquad \frac{\partial f(x,y)}{\partial y} = 3x - x^2 - 2xy$$

$$\frac{\partial f(x,y)}{\partial x} = 0 \qquad \frac{\partial f(x,y)}{\partial y} = 0$$

$$3y - 2xy - y^2 = 0 \quad \& \quad 3x - x^2 - 2xy = 0$$

$$y=0, 3-2x-y=0 \quad \& \quad x=0, 3-x-2y=0$$

Stationary points are : (0,0) , (3,0) , (0,3) , (1,1)

$$r = \frac{\partial^2 f}{\partial x^2} = -2y \quad , \quad t = \frac{\partial^2 f}{\partial y^2} = -2x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 3 - 2x - 2y$$

$$s^2 = (3 - 2x - 2y)^2$$

$$rt - s^2 = 4xy - (3 - 2x - 2y)^2$$

For point (0,0), $rt - s^2 = -9 < 0$

The point is of maxima .

For point (3,0), $rt - s^2 = -9 < 0$

The point is of maxima .

For (0,3), $rt - s^2 = -9 < 0$

The point is of maxima.

For point (1,1), $rt - s^2 = 3 > 0$

The point is of minima .

(a) Maximum values : At (0,0) , (0,3), (3,0)

At point (0,0) $f(\max)=0$

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At point (0,3) $f(\max)=0$

At point (3,0) $f(\max)=0$

(b) Minimum values : At (1,1)

At point (1,1) $f(\min)=1$

The maximum and minimum values of function are 0 and 1.

Q.3(a) Investigate for what values of μ and λ the equation $x+y+z=6$;
 $x+2y+3z=10$; $x+2y+\lambda z=\mu$ have

(i) no solution,

(ii) a unique solution,

(iii) infinite no. of solution.

[6]

Ans : Given eqn : $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$

$$A X = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array} \right]$

$R_2 - R_1,$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array} \right]$$

$R_3 - R_2,$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 3 & \mu - 10 \end{array} \right]$$

(i) When $\lambda=3, \mu \neq 10$ then $r(A) = 2, r(A : B) = 3$

$$r(A) \neq r(A : B)$$

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Hence for $\lambda=3, \mu \neq 10$ system is inconsistent.

No solution exist.

(ii) When $\lambda \neq 3, \mu \neq 10, r(A) = r(A : B) = 3$

Unique solution exist.

(iii) When $\lambda=3, \mu = 10, r(A) = r(A : B) = 2 < 3$

Infinite solution.

(b) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$. [6]

Ans : let $u = f(r, s)$

$$\therefore r = \frac{y-x}{xy} \qquad \therefore s = \frac{z-x}{xz}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} \frac{1}{x^2} + \frac{\partial u}{\partial s} \left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} = \frac{\partial u}{\partial r} \frac{(-1)}{y^2} + \frac{\partial u}{\partial s} (0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} = \frac{\partial u}{\partial r} (0) + \frac{\partial u}{\partial s} \left(\frac{1}{z^2}\right)$$

$$\therefore x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

Hence proved.

(c) Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1} \frac{b}{a}$ & $\cos [i \log\left(\frac{a+ib}{a-ib}\right)] = \frac{a^2-b^2}{a^2+b^2}$ [8]

Ans : let $L = \log\left(\frac{a+ib}{a-ib}\right)$

Using logarithmic properties ,

$$L = \log(a+ib) - \log(a-ib)$$

$$= \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \frac{b}{a} - \left[\frac{1}{2} \log(a^2 + b^2) - i \tan^{-1} \frac{b}{a} \right]$$

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$$L = 2i \tan^{-1} \frac{b}{a}$$

$$\therefore \log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1} \frac{b}{a}$$

Hence Proved.

$$\therefore \frac{a+ib}{a-ib} = e^{2i \tan^{-1} \frac{b}{a}} = \cos\left(2 \tan^{-1} \frac{b}{a}\right) + i \sin\left(2 \tan^{-1} \frac{b}{a}\right)$$

$$\frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib} = \cos\left(2 \tan^{-1} \frac{b}{a}\right) + i \sin\left(2 \tan^{-1} \frac{b}{a}\right)$$

$$\frac{a^2-b^2}{a^2+b^2} + i \text{imaginary} = \cos\left(2 \tan^{-1} \frac{b}{a}\right) + i \sin\left(2 \tan^{-1} \frac{b}{a}\right)$$

Separate real and imaginary parts

$$\cos\left(2 \tan^{-1} \frac{b}{a}\right) = \frac{a^2-b^2}{a^2+b^2}$$

From 1st result ,

$$\cos [i \log\left(\frac{a+ib}{a-ib}\right)] = \frac{a^2-b^2}{a^2+b^2}$$

Hence Proved.

Q.4(a) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Prove that

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \frac{-\sin u \cdot \cos 2u}{4 \cos^3 u} \quad [6]$$

Ans : $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put $x = xt$ and $y = yt$ to find degree.

$$\therefore u = \sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore \sin u = t^{1/2} \cdot \frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{1/2} \cdot f(x, y)$$

The function $\sin u$ is homogeneous with degree $\frac{1}{2}$.

But $\sin u$ is the function of u and u is the function of x and y .

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By Euler's theorem ,

$$xu_x + yu_y = G(u) = n \cdot \frac{f(u)}{f'(u)} = \frac{1}{2} \tan u$$

$$\begin{aligned} \therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} &= G(u)[G'(u) - 1] \\ &= \frac{1}{2} \tan u \left[\frac{\sec^2 u - 2}{2} \right] \\ &= \frac{1}{4} \tan u \left[\frac{\tan^2 u - 1}{1} \right] \\ &= \frac{1}{4} \times \frac{\sin u}{\cos u} \left[\frac{\sin^2 u - \cos^2 u}{\cos^2 u} \right] \end{aligned}$$

$$\therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \frac{-\sin u \cdot \cos 2u}{4 \cos^3 u}$$

Hence Proved.

(b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ encode and decode the message

“ALL IS WELL” .

[6]

Ans : Let encoding matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The message is ALL IS WELL and Let B is the matrix in number form,

$$B = \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

The encoded message is given by,

13 12 12 0 28 19 23 23 17 12 12 0

“MLL ASWWQLL “

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Inverse of encoding matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is given by ,

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

Decoded matrix is given by ,

$$B = A^{-1} \cdot C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$B = \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$
--

(c) Solve the following equation by Gauss Seidal method :

$$10x_1 + x_2 + x_3 = 12$$

$$2x_1 + 10x_2 + x_3 = 13$$

$$2x_1 + 2x_2 + 10x_3 = 14$$

[8]

Ans : By Gauss Seidal method ,

Given eqn : $10x_1 + x_2 + x_3 = 12$

$$2x_1 + 10x_2 + x_3 = 13$$

$$2x_1 + 2x_2 + 10x_3 = 14$$

From given eqn : $|10| > |1| + |1|$

$$|10| > |2| + |1|$$

$$|10| > |2| + |2|$$

The given eqn are in correct order.

$$\therefore x_1 = \frac{1}{10} [12 - x_2 - x_3]$$

$$\therefore x_2 = \frac{1}{10} [13 - 2x_1 - x_3]$$

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$$\therefore x_3 = \frac{1}{10} [14 - 2x_2 - 2x_1]$$

I) For 1st iteration : take $x_2 = 0, x_3 = 0$

$$x_1 = \frac{1}{10} [12] = 1.2$$

$$x_1 = 1.2, x_3 = 0 \quad \text{gives } x_2 = 1.06$$

$$x_1 = 1.2, x_2 = 1.06 \quad \text{gives } x_3 = 0.948$$

II) For 2nd iteration : take $x_2 = 1.06, x_3 = 0.948$

$$x_1 = \frac{1}{10} [12 - 1.06 - 0.948] = 0.9992$$

$$x_1 = 0.992, x_3 = 0.948 \quad \text{gives } x_2 = 1.0068$$

$$x_1 = 0.992, x_2 = 1.0068 \quad \text{gives } x_3 = 1.0002$$

III) For 3rd iteration : $x_2 = 1.0068, x_3 = 1.0002$

$$x_1 = \frac{1}{10} [12 - 1.0068 - 1.0002] = 0.9993$$

$$x_1 = 0.993, x_3 = 1.0002 \quad \text{gives } x_2 = 1.00$$

$$x_1 = 0.993, x_2 = 1.00 \quad \text{gives } x_3 = 1.00$$

Result : $x_1 = 1.00, x_2 = 1.00, x_3 = 1.00$

Q.5(a) If $u = e^{xyz} f\left(\frac{xy}{z}\right)$ where $f\left(\frac{xy}{z}\right)$ is an arbitrary function of $\frac{xy}{z}$.

Prove that : $x \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial z} = y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2xyz \cdot u$ [6]

Ans: let $\frac{xy}{z} = w \quad \therefore u = e^{xyz} \cdot f(w)$

Diff. u w.r.t. x partially,

$$\frac{\partial u}{\partial x} = e^{xyz} f'(w) + f(w) \cdot e^{xyz} \cdot yz$$

Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial y} = e^{xyz} f'(w) + f(w) \cdot e^{xyz} \cdot xz$$

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Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial z} = e^{xyz} f'(w) + f(w). e^{xyz}. xy$$

$$x \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial z} = x e^{xyz} f'(w) + f(w). e^{xyz}. xyz + z e^{xyz} f'(w) + f(w). e^{xyz}. xyz \quad \dots(1)$$

$$y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = y e^{xyz} f'(w) + f(w). e^{xyz}. xyz + z e^{xyz} f'(w) + f(w). e^{xyz}. xyz \quad \dots(2)$$

From (1) and (2),

$$x \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial z} = y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2xyz. u$$

Hence Proved.

(b) Prove that $\sin^5 \theta = \frac{1}{16} [\sin 5\theta - 5\sin 3\theta + 10\sin \theta]$ [6]

Ans : let $x = \cos \theta + i \sin \theta$ $\frac{1}{x} = \cos \theta - i \sin \theta$

$$2\cos \theta = x + \frac{1}{x} \quad \sin \theta = \frac{1}{2i} \left(x - \frac{1}{x} \right)$$

For $\sin \theta$ take fifth power on both sides ,

$$\sin^5 \theta = \left[\frac{1}{2i} \left(x - \frac{1}{x} \right) \right]^5 = \frac{1}{32i} \left[x^5 - \frac{1}{x^5} - 5 \left(x^3 - \frac{1}{x^3} \right) + 10 \left(x - \frac{1}{x} \right) \right]$$

But $x^n = \cos n\theta + i \sin n\theta$, $x^{-n} = \cos n\theta - i \sin n\theta$

$$x^n - x^{-n} = 2i \sin n\theta$$

$$\therefore \sin^5 \theta = \frac{1}{32i} [2i \sin 5\theta - 5 \times 2i \sin 3\theta + 10 \times 2i \sin \theta]$$

$$\therefore \sin^5 \theta = \frac{1}{16} [\sin 5\theta - 5\sin 3\theta + 10\sin \theta]$$

(c) i) Prove that $\log(\sec x) = \frac{1}{2} x^2 + \frac{1}{12} x^4 + \dots$

ii) Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$. [8]

Ans : (i) Let $E = \log(\sec x)$

$$= -\log(\cos x)$$

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$$= -\log \left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} \right) \right]$$

$$= - \left[- \left(\frac{x^2}{2!} - \frac{x^4}{4!} \right) - \frac{1}{2} \left(\frac{x^2}{2!} - \frac{x^4}{4!} \right) + \dots \right]$$

$$E = \log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots$$

(ii) let $f(x) = 2x^3 + 7x^2 + x - 1$

Here $a = 2$

$$f(x) = 2x^3 + 7x^2 + x - 1$$

$$f(2) = 45$$

$$f'(x) = 6x^2 + 14x + 1$$

$$f'(2) = 53$$

$$f''(x) = 12x + 14$$

$$f''(2) = 38$$

$$f'''(x) = f'''(2) = 12$$

Taylor's series is :

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

$$2x^3 + 7x^2 + x - 1 = 45 + (x - 2)53 + \frac{(x-2)^2}{2!}38 + \frac{(x-a)^3}{3!}12$$

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3$$

Q.6(a) Prove that $\sin^{-1}(\operatorname{cosec} \theta) = \frac{\pi}{2} + i \cdot \log(\cot \frac{\theta}{2})$ [6]

Ans : we have to prove this $\sin^{-1}(\operatorname{cosec} \theta) = \frac{\pi}{2} + i \cdot \log(\cot \frac{\theta}{2})$

$$(\operatorname{cosec} \theta) = \sin \left[\frac{\pi}{2} + i \cdot \log \left(\cot \frac{\theta}{2} \right) \right]$$

$$\text{R.H.S} = \sin \left[\frac{\pi}{2} + i \cdot \log \left(\cot \frac{\theta}{2} \right) \right]$$

$$= \cos \left[i \cdot \log \left(\cot \frac{\theta}{2} \right) \right] \quad \dots\dots\dots \{ \sin \left(\frac{\pi}{2} + x \right) = \cos x \}$$

$$= \cos h \log \left(\cot \frac{\theta}{2} \right) \quad \dots\dots\dots \{ \cos ix = \cos hx \}$$

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$$= \frac{1}{2} [e^{\log(\cot \frac{\theta}{2})} + e^{-\log(\cot \frac{\theta}{2})}] \quad \dots\dots\dots \{ \cos hx = \frac{1}{2} [e^x + e^{-x}] \}$$

$$= \frac{1}{2} [\cot \frac{\theta}{2} + \frac{1}{\cot \frac{\theta}{2}}]$$

$$= \frac{1}{2} \tan \frac{\theta}{2} [1 + \cot^2 \frac{\theta}{2}]$$

$$= \frac{1}{2} \tan \frac{\theta}{2} \left[\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \right] \quad \dots\dots\dots \{ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{\cot \theta} \}$$

$$= \frac{1}{2} \times \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \times \frac{1}{\sin^2 \frac{\theta}{2}}$$

$$= \frac{1}{\sin \theta}$$

$$= \operatorname{cosec} \theta = L.H.S$$

$$\therefore (\operatorname{cosec} \theta) = \sin \left[\frac{\pi}{2} + i \cdot \log \left(\cot \frac{\theta}{2} \right) \right]$$

$$\therefore \sin^{-1}(\operatorname{cosec} \theta) = \frac{\pi}{2} + i \cdot \log \left(\cot \frac{\theta}{2} \right)$$

Hence Proved.

(b) Find non singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ [6]

Ans : $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

For PAQ form ,

$$A = I_{3 \times 3} \cdot A_{3 \times 4} \cdot I_{3 \times 3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - R_1,$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + R_2,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 - C_2, C_4 - 3C_2,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-R_2,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now A is in Normal form .

Compare this w.r.t $A=PAQ$ form ,

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

\therefore Rank of given matrix A is 2.

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(c) Obtain the root of $x^3 - x - 1 = 0$ by Regula Falsi Method

(Take three iteration).

[8]

Ans : Equation : $x^3 - x - 1 = 0$

$$\therefore f(x) = x^3 - x - 1$$

$$f(0) = -1 < 0 \text{ and } f(1) = -1 < 0 \text{ and } f(2) = 5 > 0.$$

Root of given eqn lies between 1 and 2.

$$(x_0, y_0) = (1, -1) \quad (x_1, y_1) = (2, 5)$$

$$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = 1.2249$$

$$f(x_2) = -0.3871 < 0$$

Next iteration :

$$(x_0, y_0) = (1.2249, -0.3871)$$

$$(x_1, y_1) = (1.667, 1.9654)$$

$$\therefore x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = 1.2976$$

$$f(x_2) = -0.1127 < 0$$

Next iteration :

$$(x_0, y_0) = (1.2976, -0.1127)$$

$$(x_1, y_1) = (1.667, 1.9654)$$

$$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = 1.3176$$

The root of given eqn after 3rd iteration is 1.3176.

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